Restrictions of Harmonic Functions on the Sierpinski Gasket to Segments [*†]

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Abstract

The restrictions of a harmonic function on the Sierpinski Gasket (SG) to the segments in SG have been of some interest. We show that the sufficient conditions for the monotonicity of these restrictions given by Dalrymple, Strichartz and Vinson are also necessary. We then prove that the normal derivative of a harmonic function on SG on the junction points of the contour of a triangle in SG is always nonzero with at most a single exception.

We finally give an explicit derivative computation for the restriction of a harmonic function on SG to segments at specific points of the segments: The derivative is zero at points dividing the segment in ratio 1:3. This shows that the restriction of a harmonic function to a segment of SG has the following curious property: The restriction has infinite derivatives on a dense set of the segment (at junction points) and vanishing derivatives on another dense set.

1. Introduction

We will first briefly recall the rudiments of harmonic analysis on the Sierpinski Gasket. ([1],[2],[3],[4])

Let K be the Sierpinski Gasket (SG) constructed on the unit equilateral triangle G_0 with vertices $\{p_0, p_1, p_2\}$ and G_m be the graph in the m th step as in the following figure.
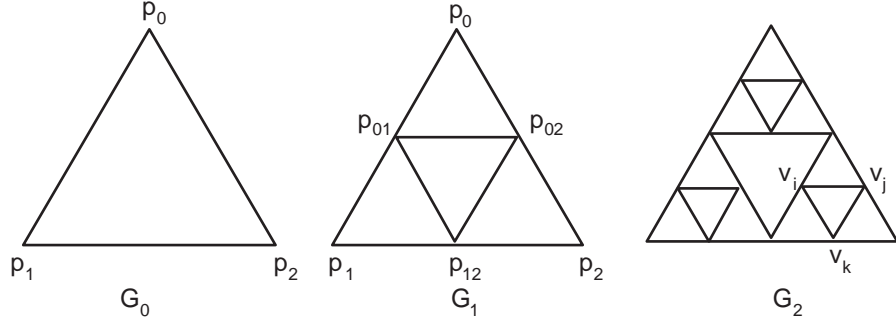

Figure 1: Iterated graphs in SG

Definition. *The function $f \in C(K)$, $f : K \longrightarrow \mathbb{R}$ is called harmonic on K if for every minimal triangle in G_m ($m \geq 1$), with vertices $\{v_i,\, v_j,\, v_k\}$ the equalities*

$$f(v_i) + f(v_j) + f(v_{ik}) + f(v_{jk}) - 4f(v_{ij}) = 0 \tag{1}$$

hold, where v_{ij} is the midpoint of the segment $[v_i, v_j]$.

Let

$$f(p_0) = \alpha,\ f(p_1) = \beta,\ f(p_2) = \gamma\,. \tag{2}$$

Then this triple (α, β, γ) completely defines a harmonic function f, that is, there exists a unique harmonic function $f : K \longrightarrow \mathbb{R}$ such that $f(p_0) = \alpha$, $f(p_1) = \beta$ and $f(p_2) = \gamma$. This harmonic function depends linearly on the triple (α, β, γ). According to the harmonic extension algorithm, it holds

$$f(p_{12}) = \frac{1}{5}(\alpha+2\beta+2\gamma), f(p_{02}) = \frac{1}{5}(2\alpha+\beta+2\gamma), f(p_{01}) = \frac{1}{5}(2\alpha+2\beta+\gamma). \tag{3}$$

We are interested in restrictions of the harmonic function f to the line segments contained in the SG. From (1)-(3) it can be seen that, if a nonconstant harmonic function is monotone on some line segment that is contained in SG, then it is strictly monotone on it.

Let T_m be a minimal triangle with vertices v_i, v_j and v_k in G_m. The sides of T_m can be ordered by the values $|f(v_i) - f(v_j)|$.

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Theorem 1 ([DSV]).
*Let f be a harmonic function on SG, E an edge in G_m with endpoints v_0, v_1
and midpoint v_{01}. Suppose*

$$f(v_0) < f(v_{01}) < f(v_1) \tag{4}$$

and

$$\frac{1}{4} \le \frac{f(v_1) - f(v_{01})}{f(v_{01}) - f(v_0)} \le 4 \, . \tag{5}$$

Then the restriction of f to E is strictly increasing.

Theorem 2 ([DSV]).
*The restriction of f to the two largest edges of T_m is monotone. On the
smallest edge of T_m, the restriction of f might be monotone or not; but if it
is not monotone, then it has a unique extremum.*
(We changed the wording of the Theorem 2 in [DSV] slightly.)

In this paper we will show that the sufficient conditions in Theorem 1 are
also necessary for the monotonicity of the restriction to an adge in G_m. Then
we will characterize when exactly simultaneous monotonicity of restrictions
to all three edges of a triangle T_m in G_m occurs. Furthermore, we will prove
that at the junction points of any segment E in G_m, the derivatives of the
restriction of a harmonic function exist improperly (possibly with exception
at a single point), and we will also prove that on another dense subset of E
the derivatives of the restriction are zero.

Now we remark that it is enough to prove these statements for the triangle
G_0 instead of considering an arbitrary triangle T_m in G_m, because the pro-
cedure of harmonic extension is the same for G_0 or T_m. In this spirit and for
simplicity we now reformulate the above two theorems of DSV (Dalrymple,
Strichartz and Vinson):

Theorem $1'$.
*Let f be a harmonic function on SG and assume $\beta = f(p_1) < \gamma = f(p_2)$.
Let p_{12} denote the midpoint of $[p_1, p_2]$. Then, the restriction of f to the edge
$[p_1, p_2]$ is strictly increasing, if the inequalities*

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i) $\beta < f(p_{12}) < \gamma$

ii) $\frac{1}{4} \le \frac{\gamma - f(p_{12})}{f(p_{12}) - \beta} \le 4$

are satisfied. (We remark that by harmonicity $f(p_{12}) = \frac{2\beta + 2\gamma + \alpha}{5}$)

Theorem $2'$.
Order the edges of G_0 by the values $|f(p_0) - f(p_1)| = |\alpha - \beta|$, $|f(p_0) - f(p_2)| = |\alpha - \gamma|$, $|f(p_1) - f(p_2)| = |\beta - \gamma|$. Then the restrictions of f to the two largest of $[p_0, p_1]$, $[p_0, p_2]$ and $[p_1, p_2]$ are monotone. On the smallest edge, the restriction of f might be monotone or not; but if it is not monotone, then it has a unique extremum.

Before proceeding further, we will recast the inequalities i) and ii) of Theorem $1'$ in a more sympatetic form:

Lemma 1.
The DSV inequalities

i) $\beta < f(p_{12}) < \gamma$

ii) $\frac{1}{4} \le \frac{\gamma - f(p_{12})}{f(p_{12}) - \beta} \le 4$

are equivalent to the inequalities

iii) $\beta < \gamma, \quad 2\beta - \gamma \le \alpha \le 2\gamma - \beta$.

(Proof is straightforward.)

2. Characterization of the Monotonicity of the Restrictions

Consider the side $[p_1, p_2] = [0,\ 1]$ of G_0 and the restriction of the harmonic function f defined by (2) to $[p_1, p_2]$. The following lemma can be proved by induction on m.

Lemma 2.
Let $l_m = \frac{1}{2} - \frac{1}{2^{m+1}}$, $r_m = \frac{1}{2} + \frac{1}{2^{m+1}}$ $(m = 1, 2, 3, ...)$. Then

$$f(\frac{1}{2^m}) = \frac{3^m - 1}{2 \cdot 5^m}\alpha + [1 - (\frac{3}{5})^m]\beta + \frac{3^m + 1}{2 \cdot 5^m}\gamma \qquad (6)$$

$$f(1 - \frac{1}{2m}) = \frac{3^m - 1}{2 \cdot 5^m}\alpha + [1 - (\frac{3}{5})^m]\gamma + \frac{3^m + 1}{2 \cdot 5^m}\beta \qquad (7)$$

$$f(l_m) = \frac{5^m - 1}{5^{m+1}}\alpha + \frac{3^{m+1} + 4 \cdot 5^m + 3}{10 \cdot 5^m}\beta + \frac{4 \cdot 5^m - 3^{m+1} - 1}{10 \cdot 5^m}\gamma \qquad (8)$$

$$f(r_m) = \frac{5^m - 1}{5^{m+1}}\alpha + \frac{3^{m+1} + 4 \cdot 5^m + 3}{10 \cdot 5^m}\gamma + \frac{4 \cdot 5^m - 3 \cdot 3^{m+1} - 1}{10 \cdot 5^m}\gamma \qquad (9)$$

(Actually, by symmetry, one of these equalities implies the other three).

Theorem 3.
Let f be the harmonic function on SG generated by the triple (α, β, γ). Then the restriction of f to $[p_1, p_2]$ is strictly increasing if and only if $\beta < \gamma$ and $2\beta - \gamma \leq \alpha \leq 2\gamma - \beta$.

Remark 1. This fact can also be expressed as follows: the restriction of a nonconstant f to $[p_1, p_2]$ is strictly monotone iff α lies between $2\beta - \gamma$ and $2\gamma - \beta$, because for a nonconstant f, necessarily $\beta \neq \gamma$: if we had $\beta = \gamma$, then α lying between $2\beta - \gamma$ and $2\gamma - \beta$ would coincide with β and γ making the function constant.
Remark 2. The condition "α lies between $2\beta - \gamma$ and $2\gamma - \beta$" can also be expressed as follows: Let $\delta = \alpha + \beta + \gamma$. Then $2\beta - \gamma - \alpha = 3\beta - \delta$, $2\gamma - \beta - \alpha = 3\gamma - \delta$ and the condition takes the form $(3\beta - \delta)(3\gamma - \delta) \leq 0$.

Proof of Theorem 3. If $\beta < \gamma$ and $2\beta - \gamma \leq \alpha \leq 2\gamma - \beta$, then by Lemma 1, the DSV inequalities are satisfied, and consequently the restriction of f to $[p_1, p_2]$ is strictly increasing by Theorem 1'. Now we show the necessity of the inequalities $\beta < \gamma$ and $2\beta - \gamma \leq \alpha \leq 2\gamma - \beta$. The inequality $\beta < \gamma$ is obvious. For all $m = 1, 2, ...$ we have by Lemma 2 and the inequality assumption

$$f(\frac{1}{2^m}) = \frac{3^m - 1}{2 \cdot 5^m}\alpha + [1 - (\frac{3}{5})^m]\beta + \frac{3^m + 1}{2 \cdot 5^m}\gamma > \beta \qquad (10)$$

$$f(1 - \frac{1}{2^m}) = \frac{3^m - 1}{2 \cdot 5^m}\alpha + [1 - (\frac{3}{5})^m]\gamma + \frac{3^m + 1}{2 \cdot 5^m}\beta < \gamma \qquad (11)$$

From (10) and (11) we obtain

$$\sup_m \frac{2 \cdot 3^m\beta - 3^m\gamma - \gamma}{3^m - 1} \leq \alpha \leq \inf_m \frac{2 \cdot 3^m\gamma - 3^m\beta - \beta}{3^m - 1} \qquad (12)$$

Computing the supremum and infimum gives

$$2\beta - \gamma \leq \alpha \leq 2\gamma - \beta \quad \square$$

Now we will find conditions on α, β and γ which guarantee the monotonicity of restrictions of a harmonic function f on SG to all three edges of G_0.

Theorem 4.
The restrictions of a nonconstant harmonic function f on SG to all three edges of the triangle G_0 are simultaneously strictly monotone if and only if one of the following conditions holds:

$$2\alpha = \beta + \gamma$$

or

$$2\beta = \alpha + \gamma$$

or

$$2\gamma = \alpha + \beta$$

Proof. By Theorem 3 and the Remarks 1 and 2, the restriction of f to all three edges of SG are simultaneously strictly monotone if and only if all of the following three inequalities are satisfied:

$$(3\beta - \delta)(3\gamma - \delta) \leq 0$$

$$(3\gamma - \delta)(3\alpha - \delta) \leq 0$$

$$(3\alpha - \delta)(3\beta - \delta) \leq 0.$$

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These three inequalities can only be satisfied if at least one of the numbers $3\alpha - \delta, 3\beta - \delta$ or $3\gamma - \delta$ vanishes. This is also sufficient as easily can be seen. For example, let $3\alpha - \delta = 0$. Then the second and third inequalities are obviously satisfied and the first is also true, because then $2\alpha = \beta + \gamma$ and this means that $\alpha = \frac{1}{2}[(2\beta - \gamma) + (2\gamma - \beta)]$ and this is enough for the first inequality to be satisfied □

Remark 3. Interestingly, the equations $2\alpha = \beta + \gamma$, $2\beta = \alpha + \gamma$, $2\gamma = \alpha + \beta$ are equivalent to the vanishing of the normal derivatives at the vertices. [BST]

Remark 4. We can express the content of Theorem 4 also in terms of side lengths of G_0 (with respect to f) : By definition, the lengths of the edges of SG are $|\alpha - \beta|, |\alpha - \gamma|, |\beta - \gamma|$. By Theorem 2 ([DSV]) we know that the restriction of f to the two largest edges are monotone. After having characterized simultaneous strictly monotonicity on all three edges in Theorem 4, we can say that this is exactly the case if and only if the G_0 is isosceles, the third side being the largest: For example, if $2\alpha = \beta + \gamma$, then $|\alpha - \beta| = |\alpha - \gamma|$ and $|\beta - \gamma| = 2|\alpha - \beta| > |\alpha - \beta|$. In other words, a non-monotone restriction occurs, iff the length of the smallest edge is strictly less than the lengths of the other two edges. So, there are two cases for non-monotonicity: Either G_0 is scalene, or , if it is isosceles, then the length of the third side is strictly less than the others. In the latter case, the third side of the isosceles must have necessarily side length zero. (See Fig. 2)

3. Derivatives of Restrictions at the Junction Points

We will show in this section that the derivatives of the restriction of a harmonic function on SG to any segment in SG exist and are infinite at all junction points with possibly a single exception. It is again enough to show this for the edges of G_0 by general reasons we indicated in the introduction.

We first need the following

Lemma 3.
Let the function $g : [0,1] \longrightarrow \mathbb{R}$ be strictly monotone in a neighborhood of $x_0 \in [0,1]$, $d \in (0,1)$, $a \neq 0$ and $x_m = x_0 + ad^m$. Assume
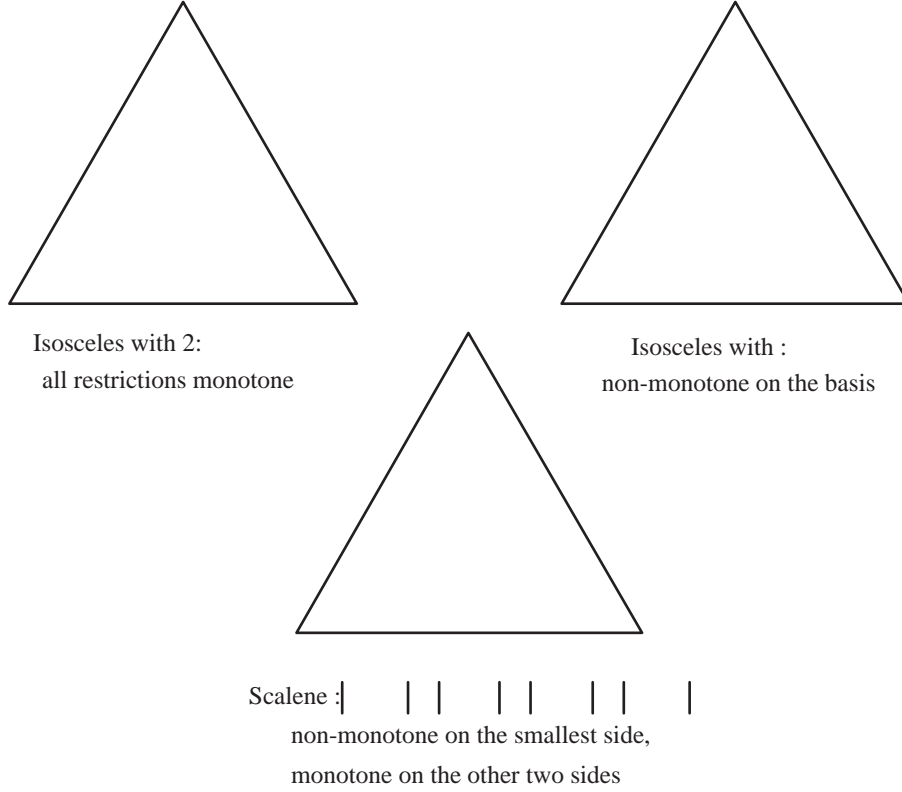
Figure 2: Monotonicity classification with respect to side-lengths of G_0

$$\frac{g(x_m) - g(x_0)}{x_m - x_0}$$

is defined and tends to 0 (or $\pm\infty$) as $m \longrightarrow \infty$. If $a < 0$ then the left derivative of g at x_0 exists and is 0 (or $\pm\infty$); If $a > 0$ then the right derivative of g at x_0 exists and is 0 (or $\pm\infty$).

Proof. We consider only the case, where g is monotone increasing and $a > 0$. Let $x_0 \in [0, 1)$ and $x > x_0$. Then there exists $m \in \mathbb{N}$ such that

$$x_0 + ad^{m+1} \leq x \leq x_0 + ad^m$$

As x tends to x_0 , m tends to infinity and from the inequalities

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$$d.\frac{g(x_{m+1}) - g(x_0)}{x_{m+1} - x_0} \leq \frac{g(x) - g(x_0)}{x - x_0} \leq \frac{1}{d} \cdot \frac{g(x_m) - g(x_0)}{x_m - x_0}$$

we get the result $\square$

Remark 5. In the above lemma, one-sided monotonicity is obviously enough for one-sided derivative calculations.

Now we will compute the derivative of the restriction at the point $p = 1/2$, for monotone restrictions.

Lemma 4.
Let the restriction of the harmonic function f to the edge $[p_1, p_2] = [0, 1]$ be strictly monotone. Then $f'(\frac{1}{2}) = +\infty$ for f monotone increasing and $f'(\frac{1}{2}) = -\infty$ for f monotone decreasing.

Proof. We give the proof for f monotone increasing:
Applying (8) and Lemma 1, where $x_0 = \frac{1}{2}$, $a = -\frac{1}{2}$, $d = \frac{1}{2}$, we obtain

$$\lim_{m \to \infty} \frac{f(l_m) - f(\frac{1}{2})}{l_m - \frac{1}{2}} = \lim_{m \to \infty} (\frac{3}{5}).(\frac{6}{5})^m (\gamma - \beta) = +\infty.$$

Then by Lemma 3 the left-hand derivative at $p = \frac{1}{2}$ is $+\infty$. Analogously, using (9), we obtain that the right-hand derivative at $p = \frac{1}{2}$ is also $+\infty$ $\square$

Applying Lemma 4 to smaller triangles, we see that the derivatives exist improperly at all inner junction points of $[p_1, p_2]$ in whose vicinity the restriction is strictly monotone. Using Lemma 3 and Lemma 2, (6) and (7), we can compute the derivatives at p_1 and p_2 also.

Lemma 5.
If $2\beta = \alpha + \gamma$, then the derivative of the restriction of f to $[p_1, p_2]$ at p_1 is zero; otherwise infinite. Similarly, if $2\gamma = \alpha + \beta$, then the derivative of the restriction at p_2 is zero, otherwise infinite.
(This result is implicit in Theorem 4 of [DSV]).

Lemma 4 and 5 yield to

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Lemma 6.

Let f be a non-constant harmonic function on SG. Then the derivatives of the restriction of f to $[p_1, p_2]$ exist improperly at all junction points on $[p_1, p_2]$, with possibly a single exception. Moreover, this is true for the whole contour $G_0 = [p_0, p_1] \cup [p_0, p_2] \cup [p_1, p_2]$, i.e. the derivative of the restrictions can vanish only for a single junction point on the whole contour and is infinite for all the junction points.

The derivatives of the restriction are related to normal derivatives (For normal derivatives see [BST]). It can be seen from the work of [BST] using monotonicity of restrictions of a harmonic function f on SG, that if the normal derivative at a junction point vanishes, then the derivative of the restriction to a segment containing the junction point also vanishes at that junction point. But we have seen above, that at junction points on the contour G_0 the derivative is infinite with possibly a single exception. This proves that the normal derivative is nonzero for all junction points on the contour G_0 with at most a single exception for a non-constant f.

Remark 6. It can be shown that, if the numbers α, β, γ (being not all equal) are linearly independent over the field of rational numbers, then the normal derivative of the harmonic function on SG determined by α, β, γ is never zero on any junction point on SG. More strictly, if there does not exist a relation

$$n\alpha + m\beta + k\gamma = 0$$

between α, β, γ (being not all equal) with n, m, k integers and $n + m + k = 0$, then the normal derivative is never zero on any junction point on SG.

Summarizing the above considerations, we obtain

Theorem 5.

The normal derivative of a non-constant harmonic function f on SG is non-zero at all junction points on $G_0 \subset SG$, with at most a single exception. This exception occurs at a vertex of G_0, iff all restrictions of f to the edges of G_0 are monotone.

4. Zero Derivatives

In this section we will show that the derivative of the restriction of a harmonic function f on SG to an edge of any G_m is differentiable at a point

dividing the edge in ratio 1:3 and the derivative there vanishes. It is again enough to show this for the edge $[p_1, p_2] = [0, 1]$ of G_0 as the extention rule for the harmonic function is the same at every scale.

Theorem 6.
Let f be a harmonic function on SG and p the point dividing the edge $[p_1, p_2]$ in ratio 1:3. (i.e. $p = 1/3$). Then

$$(f|_{[0,1]})'(\frac{1}{3}) = 0.$$

Proof. Let us first assume that the restriction of f to $[0, 1]$ is monotone increasing. To approach the point $p = \frac{1}{3}$ from left and right with geometrically convergent sequences we use the following sequence of triangles $\triangle_m = \{p_0^m, p_1^m, p_2^m\}$:

Let $\triangle_0 = G_0 = \{p_0, p_1, p_2\}$ and let $\triangle_m$ be defined as in Fig 3. (right third of the left third of $\triangle_{m-1}$)
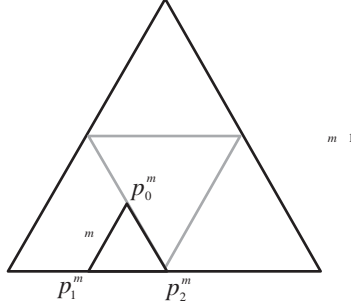

Figure 3: The sequence of triangles $\triangle_m$

One can compute

$$p_1^m = \frac{1}{4} + (\frac{1}{4})^2 + \dots + (\frac{1}{4})^m = \frac{1}{3} - \frac{1}{3}(\frac{1}{4})^m$$

$$p_2^m = p_1^m + (\frac{1}{4})^m = \frac{1}{3} + \frac{2}{3}(\frac{1}{4})^m.$$

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Let $f(p_0^m) = \alpha_m$, $f(p_1^m) = \beta_m$ and $f(p_2^m) = \gamma_m$, $\alpha_0, \beta_0, \gamma_0$ being α, β, γ.

We want to compute the values β_m and γ_m explicitly. Using (3) we get

$$\alpha_m = \frac{1}{25}[6\alpha_{m-1} + 13\beta_{m-1} + 6\gamma_{m-1}] \tag{13}$$

$$\beta_m = \frac{1}{25}[4\alpha_{m-1} + 16\beta_{m-1} + 5\gamma_{m-1}] \tag{14}$$

$$\gamma_m = \frac{1}{5}[\alpha_{m-1} + 2\beta_{m-1} + 2\gamma_{m-1}] \tag{15}$$

From (13)-(15) we obtain

$$5\alpha_m + 15\beta_m + 7\gamma_m = 5\alpha_{m-1} + 15\beta_{m-1} + 7\gamma_{m-1}$$

for all $m = 1, 2, ...$ In other words,

$$5\alpha_m + 15\beta_m + 7\gamma_m = 5\alpha + 15\beta + 7\gamma =: c. \tag{16}$$

From (16) and continuity of f we get

$$f(\frac{1}{3}) = \frac{c}{27}. \tag{17}$$

Using (16) we can eliminate α_{m-1} from (15):

$$\beta_m = \frac{1}{125}[4c + 20\beta_{m-1} - 3\gamma_{m-1}] \tag{18}$$

$$\gamma_m = \frac{1}{125}[5c - 25\beta_{m-1} + 15\gamma_{m-1}] \tag{19}$$

As can be seen from (18) and (19), the sequence

$$t_m = u\beta_m + v\gamma_m \tag{20}$$

with $u = 10$, $v = 1 - \sqrt{13}$, satisfies the recursion formula

$$t_m = w + st_{m-1}, \tag{21}$$

where $w = \frac{9-\sqrt{13}}{25}c$, $s = \frac{7+\sqrt{13}}{50}$.

From (21) t_m can be determined:

$$t_m = w\frac{s^m - 1}{s - 1} + s^m.t_0 \qquad (t_0 = 10\beta + (1 - \sqrt{13})\gamma). \qquad (22)$$

From (18), (19) and (21) we obtain

$$\gamma_m = \frac{c}{25} - \frac{1}{50}t_{m-1} + \frac{v + 6}{50}\gamma_{m-1},$$

and inserting t_{m-1} from (22) we get

$$\gamma_m = l + k.s^{m-1} + h.\gamma_{m-1}, \qquad (23)$$

where $l = \frac{c}{25} + \frac{w}{50(s-1)}$, $\quad k = -\frac{1}{50}(\frac{w}{s-1} + t_0)$, $\quad h = \frac{v+6}{50}$.

The recursion (23) gives γ_m explicitly:

$$\gamma_m = [\frac{l}{h - 1} - \frac{k}{s - h} + \gamma]h^m + \frac{k}{s - h}s^m + \frac{c}{27}.$$

As $0 < h < \frac{1}{4}$ and $0 < s < \frac{1}{4}$ we obtain finally

$$\lim_{m \longrightarrow \infty} \frac{f(p_2^m) - f(\frac{1}{3})}{p_2^m - \frac{1}{3}} = 0.$$

Taking $x_0 = \frac{1}{3}$, $d = \frac{1}{4}$ and $a = \frac{2}{3}$ in Lemma 3, we see that the right derivative of the restriction of f to $[p_1, p_2] = [0, 1]$ at $p = 1/3$ exists and is zero.

Similarly, from (18), (19), (21) we get

$$\beta_m = \frac{1}{u}[\frac{w}{s - 1} - \frac{vk}{s - h} + t_0].s^m - \frac{v}{u}(\frac{l}{h - 1} - \frac{k}{s - h} + \gamma).h^m + \frac{c}{27}$$

and this shows that the left derivative at $p = \frac{1}{3}$ exists and is also zero. Together we obtain

$$(f|_{[0,1]})'(\frac{1}{3}) = 0.$$

Now we consider the case where the restriction of f to $[0, 1]$ is not monotone. In that case we know that the restriction is monotone in two pieces. If the extremum is not attained at $p = 1/3$, then there is a neighborhood $(\frac{1}{3} - \delta, \frac{1}{3} + \delta)$ where the restriction is monotone and the above proof applies.

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If the extremum is attained at $p = 1/3$, then Lemma 2,3 and the above proof works still on two sides of $p = 1/3$ and we get $(f|_{[0,1]})'(\frac{1}{3}) = 0$ □

References

[1] Ben-Bassat, O., Strichartz, R. and Teplyaev, A. (1999). What is not in the domain of the Laplacian on a Sierpinski gasket type fractal, *J. Functional Anal.*,**166**, 197-217.

[2] Dalrymple, K., Strichartz, R. and Vinson, J. (1999). Fractal Differential Equations on the Sierpinski Gasket, *J. Four. Anal. Appl.*,**5**, 203-284.

[3] Kigami, K. (2001). Analysis on Fractals, *Cambridge University Press.*

[4] Yamaguti, M., Hata, M. and Kigami, J. (1997). Mathematics of Fractals, *Amer. Math. Soc.*

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